ISTITUTO ITALIANO DI TECNOLOGIA COMPUTATIONAL STATISTICS AND MACHINE LEARNING
ISTITUTO ITALIANO DI TECNOLOGIA DYNAMIC LEGGED SYSTEMS

# Dynamics Harmonic Analysis of Robotic Systems

## Application in Data-Driven Koopman Modelling

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## Symmetries in Dynamics

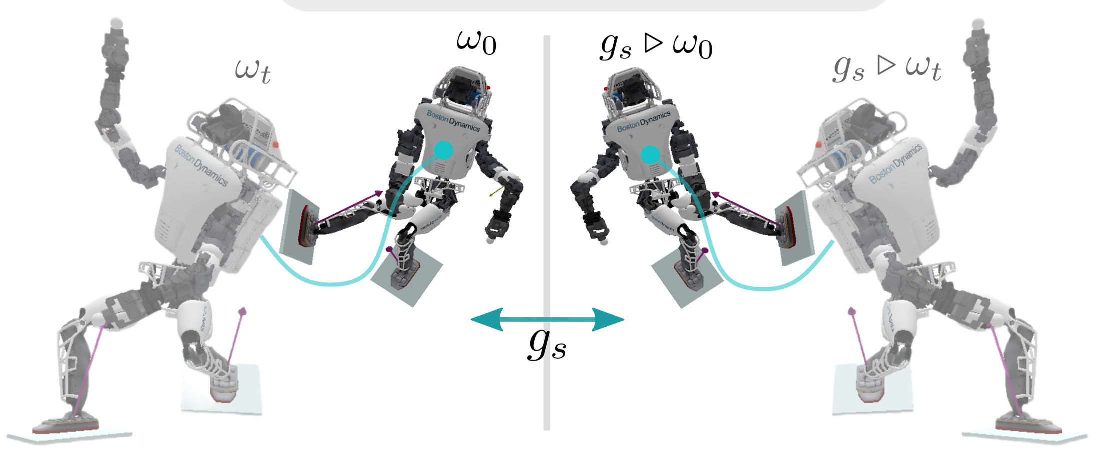
We study the problem of learning models of the dynamics of systems with finite state symmetry groups.

**Symmetric Dynamical System:** Let $\Omega$ be the set of system's states and $\Phi^{\Delta t}_{\Omega}: \Omega \mapsto \Omega$ its evolution map, such that the system's dynamics are given by $\omega_{t+\Delta t} := \Phi^{\Delta t}_{\Omega}(\omega_t)$. Such systems are said to posses a state symmetry group $\mathbb{G}$ if their evolution map is equivariant, that is:

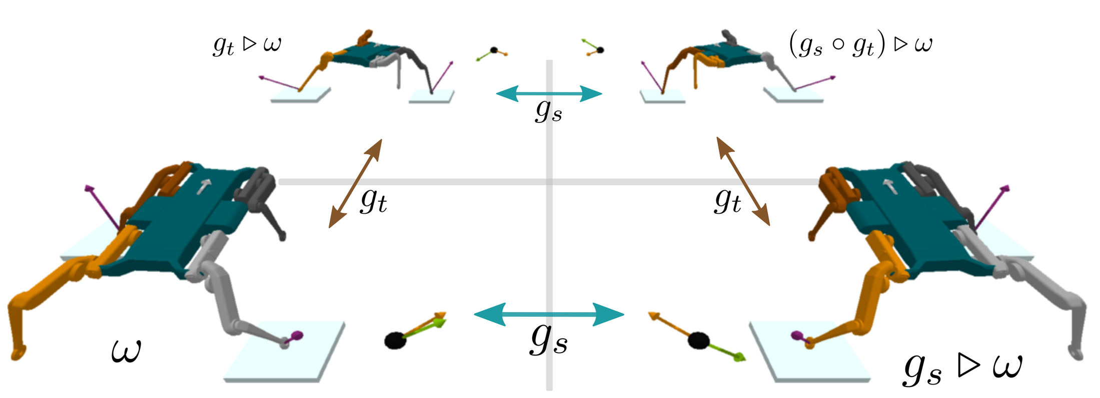
$$g \triangleright \omega_{t+\Delta t} = \Phi^{\Delta t}_{\Omega}(g \triangleright \omega_t), \quad \forall\, g \in \mathbb{G}, \omega_t \in \Omega$$

Such systems are ubiquitous in physics, robotics, and computer graphics.

$\mathbb{G} \cong \mathbb{C}_2 = \{e, g_s | g_s \circ g_s = e\}$



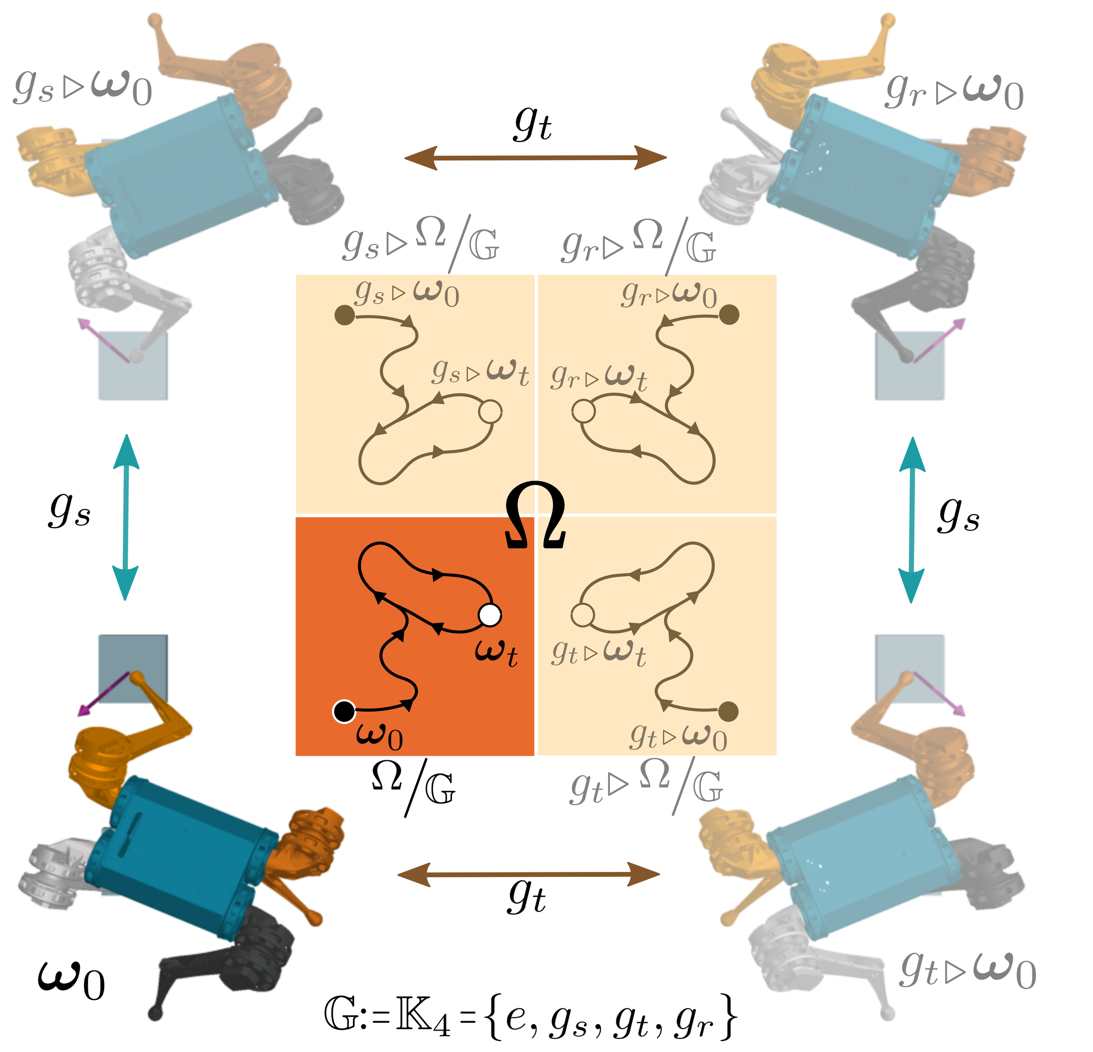

$\mathbb{G} \cong \mathbb{K}_4 = \{e, g_s, g_t, g_r | g_s^2 = g_t^2 = g_r^2 = e, g_s \circ g_t = g_r\}$




## Modelling symmetric systems

When designing numerical model of a system's dynamics we normally select a set of **observable functions** $\{x_1, \dots, x_m \mid x_i : \Omega \mapsto \mathbb{R}\}$, which define a state resentation function $\boldsymbol{x} := [x_1, \dots, x_m] : \Omega \mapsto \mathcal{X}$ along with a modeling state space $\mathcal{X} \subseteq \mathbb{R}^m$, and an approximate evolution map $\Phi^{\Delta t}_{\mathcal{X}} : \mathcal{X} \mapsto \mathcal{X}$.

For symmetric dynamical systems we are interested in models that exploit the symmetric structure of the set of states $\Omega$. This structure stems from the relationship between each state $\omega \in \Omega$ and its set of symmetric states $\mathbb{G}\omega = \{g \triangleright \omega \mid \forall\, g \in \mathbb{G}\}$.




**Symmetric Model:** A model of a symmetric dynamical systems is denoted as symmetric if:

- The modeling space $\mathcal{X} \subseteq \mathbb{R}^m$ is a **symmetric vector space.**
- The state representation funtion $\boldsymbol{x} : \Omega \mapsto \mathcal{X}$ is $\mathbb{G}$**-equivariant**
- The approximate evolution map is $\mathbb{G}$**-equivariant.**

$$\begin{array}{ccc} \omega & \xrightarrow{g \in \mathbb{G}} & g \triangleright \omega \\ \downarrow \boldsymbol{x} & & \downarrow \boldsymbol{x} \\ \boldsymbol{x}(\omega) & \xrightarrow{g \in \mathbb{G}} & g \triangleright \boldsymbol{x}(\omega) \end{array}$$

$$g \triangleright \Phi^{\Delta t}_{\mathcal{X}}(\boldsymbol{x}(\omega_t)) = \Phi^{\Delta t}_{\mathcal{X}}(\boldsymbol{x}(g \triangleright \omega_t)), \quad \forall\, g \in \mathbb{G}, \omega_t \in \Omega$$

## Harmonic analysis of symmetric dynamics' models

For symmetric models of dynamical systems with **finite symmetry groups**, the structure of the symmetric modeling space $\mathcal{X} \subseteq \mathbb{R}^m$ can be leveraged through the isotypic decomposition, a standard result from harmonic analysis stating that there exist a change of basis $\boldsymbol{T} : \mathcal{X} \mapsto \mathcal{X}$ which exposes the **orthogonal decomposition** of the modeling space into a finite number of isotypic-subspaces, each featuring a unique subgroup of symmetries:

$$\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2 \times \cdots \times \mathbb{G}_k, \quad |\mathbb{G}| < \infty$$

Number of unique types of Irreps of $\mathbb{G}$

Symmetry **subgroup** associated to irrep of type 1

$$\mathcal{X} = \mathcal{X}_1 \oplus^{\perp} \mathcal{X}_2 \oplus^{\perp} \cdots \oplus^{\perp} \mathcal{X}_k$$

Iso-subspace with symmetry subgroup $\mathbb{G}_k < \mathbb{G}$

Isotypic subspace with symmetry subgroup $\mathbb{G}_1 < \mathbb{G}$

$$\boldsymbol{x}(\omega) := \boldsymbol{T}^{-1} \begin{bmatrix} \boldsymbol{x}^{(1)}(\omega) \in \mathcal{X}_1 \\ \boldsymbol{x}^{(2)}(\omega) \in \mathcal{X}_2 \\ \vdots \\ \boldsymbol{x}^{(k)}(\omega) \in \mathcal{X}_k \end{bmatrix}$$

Projection of $\boldsymbol{x}(\omega)$ in the isotypic subspace $\mathcal{X}_k$

State representation in $\mathcal{X} \subseteq \mathbb{R}^m$

Example: Linear system in 3D

Let $\mathcal{X} \subseteq \mathbb{R}^3$ and $\Phi^{\Delta t}_{\mathcal{X}} := \boldsymbol{A} \in R^{3\times 3}$ such that:

- $\operatorname{eig}(\boldsymbol{A}) = [a + \mathrm{i}\, b, a - \mathrm{i}\, b, c], \quad a, b, c \in \mathbb{R}$
- $\mathbb{G} \cong \mathbb{C}_3 = \{e, g_1, g_1^2 | g_1^3 = e\}$

Then, the isotypic decomposition of $\mathcal{X}$ reduces to a change of basis $\boldsymbol{T} : \mathcal{X} \mapsto \mathcal{X}$ exposing:

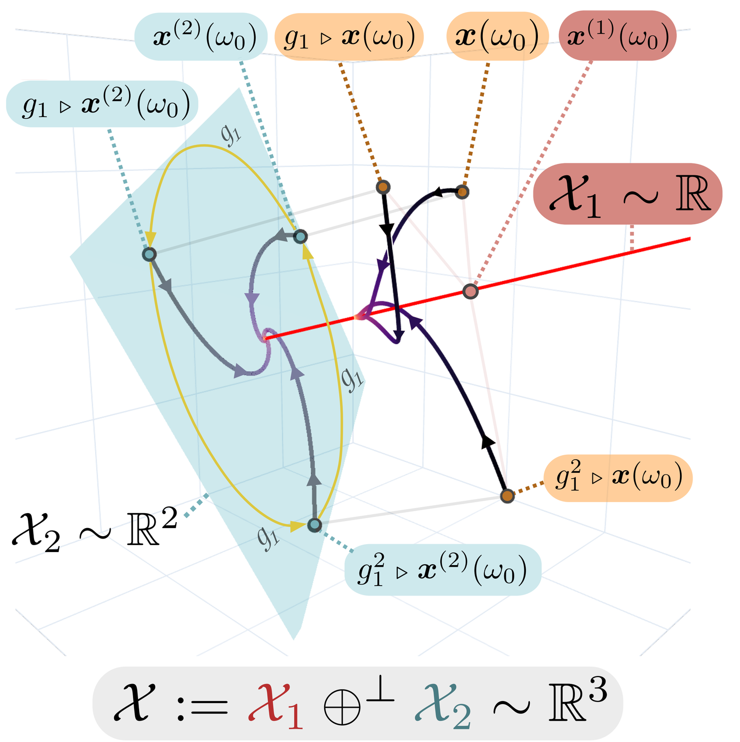
Line with subgroup $\mathbb{G}_1 = \{e\}$

$\mathcal{X} = \mathcal{X}_1 \oplus^{\perp} \mathcal{X}_2$

Plane with group $\mathbb{G}_2 \cong \mathbb{C}_3$

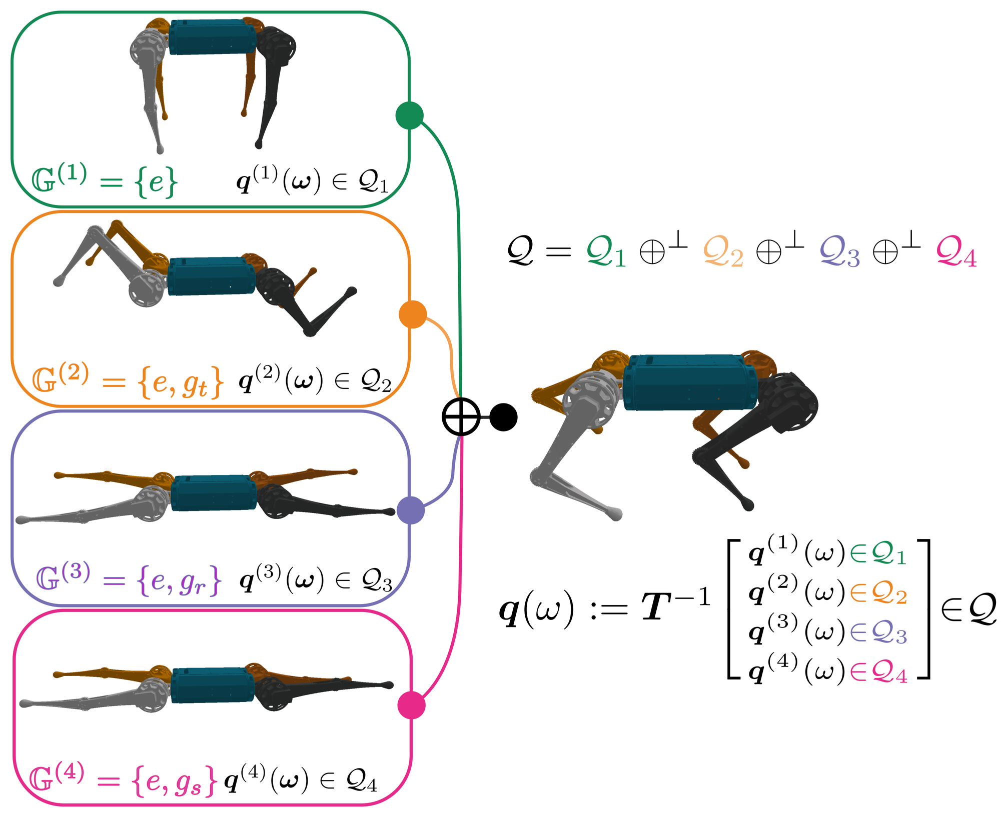
$\boldsymbol{x}(\omega) = \boldsymbol{T}^{-1}(\boldsymbol{x}^{(1)}(\omega) \oplus \boldsymbol{x}^{(2)}(\omega))$

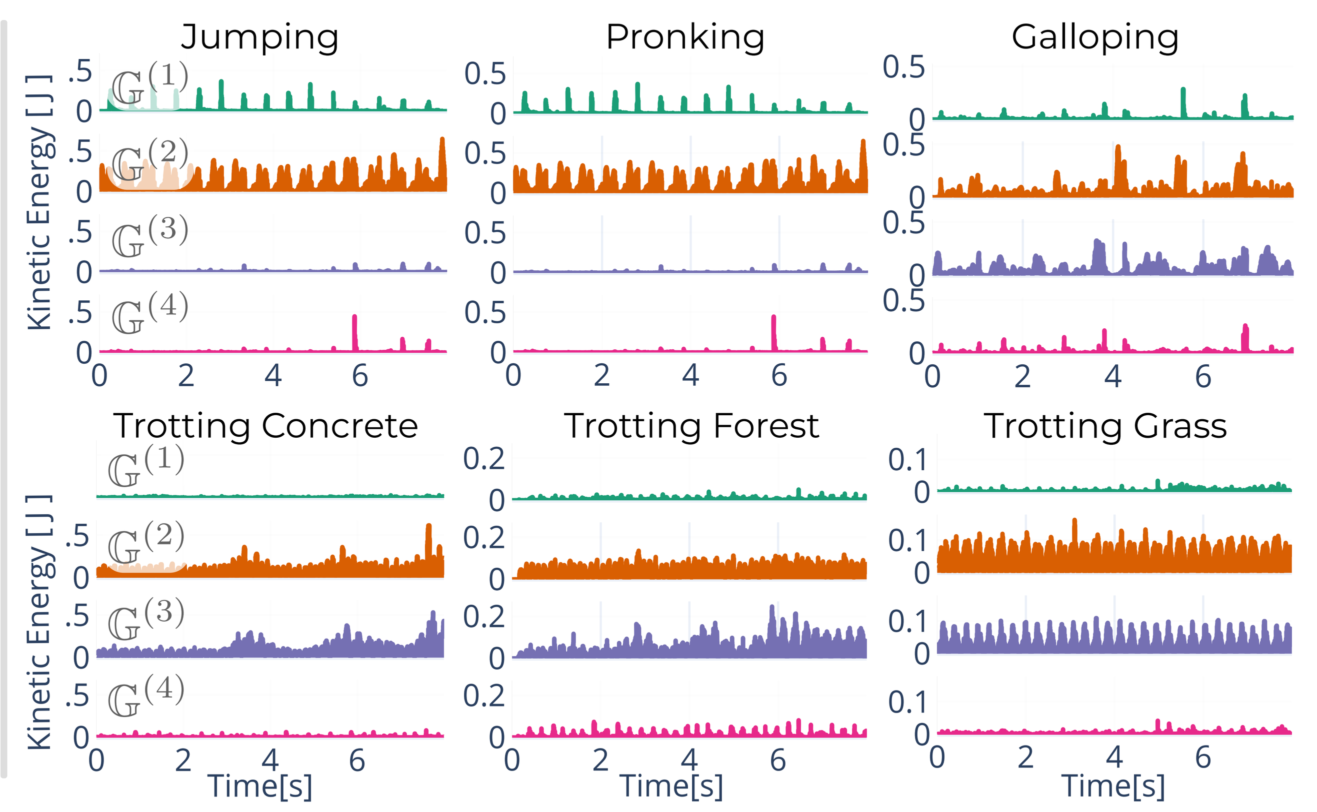
Projection of state in 2D plane $\mathcal{X}_2$




## Harmonic analysis of the space of generalized coordinates

When applying the isotypic decomposition to the space of position generalized coordinates of symmetric robotic systems, the resultant isotypic subspaces characterize orthogonal spaces of state configurations. These subspaces become relevant in characterizing the lower-dimensional structure of locomotion gaits.




## Harmonic analysis of the space of observable functions

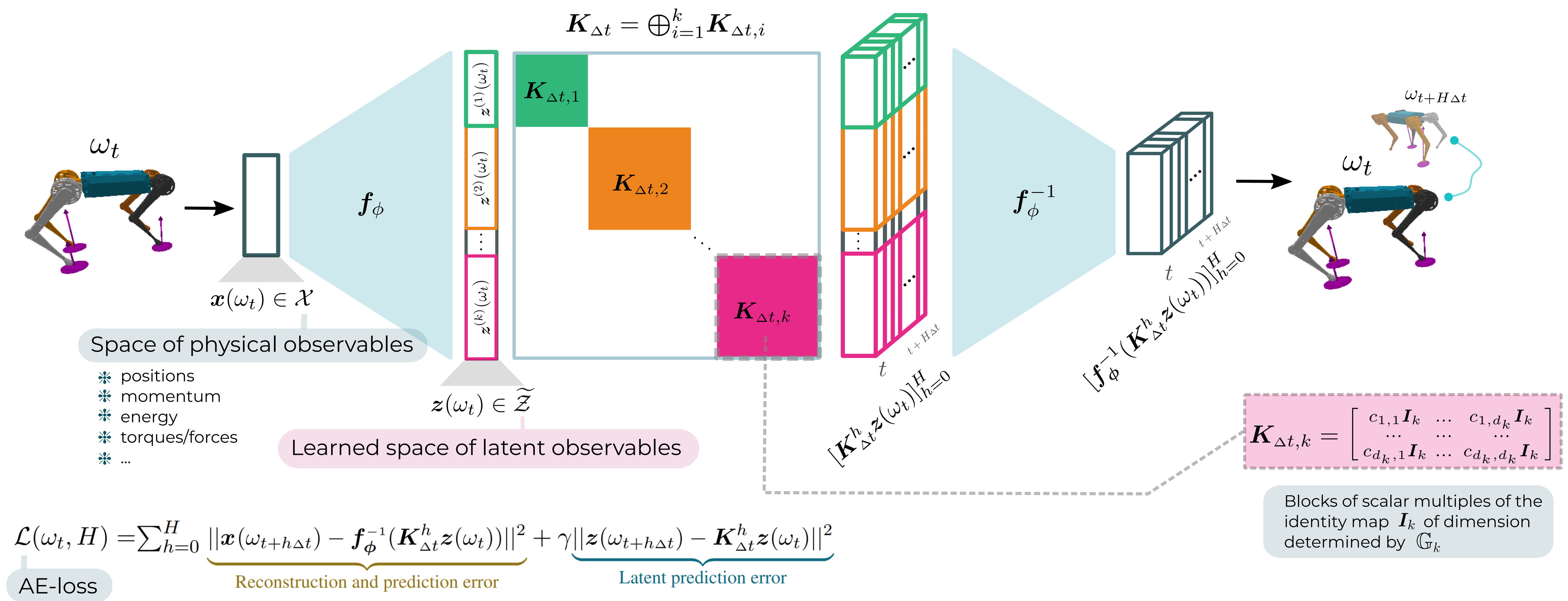
Our work leverages the geometric prior of state symmetries (and the isotypic decomposition) in dynamics models based on Koopman operator theory. These models capture temporal dynamics using the Koopman linear operator $\mathsf{K}_{\Delta t} : \mathcal{F}_{\mathcal{Z}} \mapsto \mathcal{F}_{\mathcal{Z}}$ in the infinite-dimensional space of **all** state observable functions $\mathcal{F}_{\mathcal{Z}} = \{z : \Omega \mapsto \mathbb{R}\}$, such that:

$$[\mathsf{K}_{\Delta t}\, z](\omega_t) := z(\Phi^{\Delta t}_{\Omega}(\omega_t)) = z(\omega_{t+\Delta t}), \ z \in \mathcal{F}_{\mathcal{Z}} : \Omega \to \mathbb{R}, \ \omega \in \Omega.$$

To approximate these models in finite dimensions one needs to learn a (latent) state representation function $\boldsymbol{z} = [z_1, \dots, z_\ell] : \Omega \to \widetilde{\mathcal{Z}} \subseteq \mathbb{R}^\ell$, that spans a finite-dimensional space of functions $\mathcal{F}_{\widetilde{\mathcal{Z}}} := \{z_{\boldsymbol{\alpha}}(\cdot) := \langle \boldsymbol{z}(\cdot), \boldsymbol{\alpha} \rangle, \mid \boldsymbol{\alpha} \in \mathbb{R}^\ell\}$ on which the Koopman operator is approximated by a matrix as: $(\mathsf{K}_{\Delta t} z_{\boldsymbol{\alpha}})(\cdot) \approx z_{\boldsymbol{K}^*_{\Delta t}\boldsymbol{\alpha}}(\cdot) := \langle \boldsymbol{z}(\cdot), \boldsymbol{K}^*_{\Delta t}\boldsymbol{\alpha} \rangle = \langle \boldsymbol{K}_{\Delta t}\boldsymbol{z}(\cdot), \boldsymbol{\alpha} \rangle$

## Equivariant Dynamics Autoencoder (eDAE)

To learn Koopman operator models of symmetric dynamical systems, we present the eDAE architecture: a fully differentiable $\mathbb{G}$-equivariant auto-encoder that leverages the block-diagonal and equivariant properties of the Koopman operator.




$$\mathcal{L}(\omega_t, H) = \sum_{h=0}^{H} \underbrace{||\boldsymbol{x}(\omega_{t+h\Delta t}) - \boldsymbol{f}_\phi^{-1}(\boldsymbol{K}^h_{\Delta t}\boldsymbol{z}(\omega_t))||^2}_{\text{Reconstruction and prediction error}} + \gamma \underbrace{||\boldsymbol{z}(\omega_{t+h\Delta t}) - \boldsymbol{K}^h_{\Delta t}\boldsymbol{z}(\omega_t)||^2}_{\text{Latent prediction error}}$$

AE-loss

## Experimental results